Random Euclidean coverage and connectivity problems

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Consider sample of n uniform random points in a bounded region A in R^d, $d \geq 2$, having a smooth boundary. The *coverage threshold* T_n is the smallest r such that the union Z of Euclidean balls of radius r centred on the sample points covers A. The *connectivity threshold* K_n is twice the smallest r required for Z to be connected. The *two-sample coverage threshold* $S_{n,m}$ is the smallest r such that Z covers all the points of a second independent sample of m points in A. These thresholds are random variables determined by the sample, and are of interest, for example, in wireless communications, set estimation, and topological data analysis.

We discuss new/recent results on the large-n limiting distributions of T_n, and K_n and $S_{n,m}$ (taking $m = m(n) \sim \tau n$ for some constant τ). When A has unit volume, with v denoting the volume of the unit ball in R^d and $|dA|$ the perimeter of A, these take the form of weak convergence of $nvT_n^d - (2 - 2/d)\log n - a_d \log(\log n)$ to a Gumbel-type random variable with cumulative distribution function

$$F(x) = \exp(-b_d e^{-x} - c_d |dA| e^{-x/2}),$$

for suitable constants a_d, c_d with $b_2 = 1$, $b_d = 0$ for $d > 2$. The corresponding result for K_n takes the same form with different constants a_d, c_d.

If time permits, we may also discuss extensions and related results concerning (i) taking A to be a polytope rather than having a smooth boundary; (ii) taking A to be a d-dimensional manifold with boundary embedded in a higher-dimensional Euclidean space; (iii) strong laws of large numbers for T_n and K_n for non-uniform random samples of points.

Some of the work described here is joint work with Xiaochuan Yang.

[1] Penrose, M.D. (2023) Random Euclidean coverage from within. *Probab. Theory Rel. Fields* **185**, 747–814.