Beyond classical Exponential Analysis: Generalizations, Connections and Applications.

Exponential analysis might sound remote, but it touches our lives in many surprising ways [8], even if most people are unaware of just how important it is. For example, exponential functions with real exponents are used to portray relaxation, chemical reactions, radioactivity, fluid dynamics, heat transfer. A substantial amount of effort in the field of signal processing is essentially dedicated to the analysis of exponential functions whose exponents are complex. Moreover, the analysis of exponential functions whose exponents are very near each other is directly linked to superresolution, which is a hot topic in both 1-d signal processing and 2-d or 3-d image processing [2, 10].

The classical problem of exponential analysis in one variable is a nonlinear data fitting problem. It is termed an inverse problem because it consists in extracting an exponential model's linear and nonlinear parameters from a limited number of observations of the model's behaviour. A discrete eigenfunction property of the exponential function allows to split the problem into two parts, one of which is delivering the nonlinear parameters as generalized eigenvalues and the other one the linear coefficients from a structured linear system. Input to both the generalized eigenvalue problem and the structured linear system are samples taken at equidistant points in the domain. Inverse problems have a tendency to be numerically sensitive, but as explained below, a possibility exists to deal with this.

Since exponential models are vital in the description of physical as well as biological phenomena, their analysis is crucial. Though several generalizations of exponential analysis have been studied in the past decades [9, 7, 1, 11, 12, 14, 13], a full-blown generalization of the theory is lacking and the multivariate problem statement is far less understood and developed. The common bottleneck of multidimensional problems is that when the dimensionality increases, the amount of data needed to support a reliable result often grows exponentially with the dimensionality. This phenomenon is also referred to as the curse of dimensionality. For example, 100 evenly spaced sample points suffice to sample a unit interval uniformly with a distance of 0.01 between the points. However, an equivalent sampling of a 10-dimensional unit hypercube with a lattice spacing of only 0.01 between adjacent points requires 10^{20} or a gazillion number of sample points.

Only in 2017, the UAntwerpen (UA) research group "Computational Mathematics (CMA)" developed a method for exponential analysis which does not suffer from the curse of dimensionality [6]. It can actually solve the multidimensional problem from the theoretical minimum number of samples if the data are noisefree. Another great (EU and US patented) UA-CMA breakthrough is the ability to recover signals from sub-Nyquist sampling [4, 5], meaning from much coarser sampling than requested by the fundamental Nyquist theorem, thereby opening up a whole new world of applications while improving the numerical conditioning of this inverse problem.

Exponential analysis is closely related to tensor decomposition in multilinear algebra, sparse interpolation from computer algebra, and Padé approximation from rational approximation theory [3]. Despite the fact that these seemingly unrelated and diverse topics are very intertwined, many connections remain unexplored.

We are confident that a combination of these novel possibilities can provide essential contributions to antenna design, seismic data processing, peak fitting in analytical chemistry, direction of arrival problems, radar imaging, the marine and transportation sectors, trajectory optimisation in repetitive tasks, and much more. We have therefore started up research projects in the mentioned application domains.

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